M E T U
Department of Mathematics

Group	CALCULUS II MidTerm 1	Exam Place.

Code : *Math 120* Acad. Year : *2007-2008* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *April.12.2008* Time : *09:30* Duration : *100 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 60 POINTS

1	2	3	4	5	6	Show Your Work	

Question 1 (10 points) Find the interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{(n-1)\,9^n}$ (Do not forget to check the end points.)

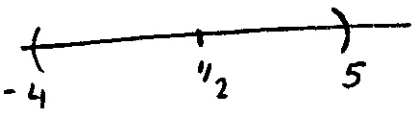
Apply Ratio Test: $\lim_{n\to\infty} \left| \frac{(2x-1)^{n+1}}{n \cdot 9^{n+1}} \cdot \frac{(n-1)\,9^n}{(2x-1)^n} \right| = \frac{|2x-1|}{9} \lim_{n\to\infty} \frac{n-1}{n}$

$= \frac{|2x-1|}{9}$

Thus given power series around $a = \frac{1}{2}$ is (absolutely) convergent if $\frac{|2x-1|}{9} < 1$ That is if $-9 < 2x-1 < 9$

$-8 < 2x < 10$

$-4 < x < 5$



ie, Radius of convergence $R = \frac{9}{2}$

Now check the end points:

$x = -4 \Rightarrow \sum_{n=2}^{\infty} \frac{(-9)^n}{(n-1)\,9^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} = a_n$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n-1} = 0$

$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{n}}{\frac{1}{n-1}} = \frac{n-1}{n} < 1 \Rightarrow |a_n| \searrow$

Hence by Alternating series test, it is convergent

$x = 5 \Rightarrow \sum_{n=2}^{\infty} \frac{9^n}{(n-1)\,9^n} = \sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent harmonic series

Thus interval of convergence is $I = [-4, 5)$

Question 2 (5+5=10 points) Test the following series for convergence:

a) $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n}+n^2+5} = \sum a_n \; ; \; a_n > 0$

$$\lim_{n\to\infty} \frac{\frac{n+1}{\sqrt{n}+n^2+5}}{\frac{1}{n}} = \lim_{x\to\infty} \frac{x^2+x}{x^2+\sqrt{x}+5} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\frac{1}{x}\right)}{\cancel{x^2}\left(1+\frac{1}{x^{3/2}}+\frac{5}{x^2}\right)} = 1$$

since limit is a nonzero finite number, $\sum a_n$ and $\sum \frac{1}{n}$ either both converge or both diverge. (by Limit Comparison Test)

Since $\sum \frac{1}{n}$ is divergent, so is $\sum a_n$.

b) $\sum_{n=1}^{\infty} \frac{2^n n!}{5\cdot 8\cdot 11\cdots(3n+2)} = \sum a_n \; ; \; a_n > 0$

Apply Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}\cdot(n+1)!}{5\cdot 8\cdot 11\cdots(3n+2)\cdot(3n+5)} \cdot \frac{5\cdot 8\cdots(3n+2)}{2^n\cdot n!}$

$= \lim_{n\to\infty} \frac{2(n+1)}{(3n+5)} = \frac{2}{3} < 1$, hence by Ratio Test, $\sum a_n$ is converge

Question 3 (5+5=10 points)

a) Prove that if each term $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent then the series $\sum_{n=1}^{\infty} a_n^2$ is also convergent.

Since $\sum a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$, hence for sufficiently large n-values, we have $0 < a_n < 1 \;\; \forall n \geq N$. $\Rightarrow a_n^2 < a_n \;\; \forall n \geq N$

$\Rightarrow 0 \leq \sum_{n=N}^{\infty} a_n^2 \leq \underbrace{\sum_{n=N}^{\infty} a_n}_{\text{conv}}$ $\Rightarrow$ by comparison test we have $\sum_{n=N}^{\infty} a_n^2$ is convergent, and hence $\sum_{n=1}^{\infty} a_n^2$ is convergent.

OR
since $a_n > 0$, we can apply Limit Comp. Test
$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n = 0$ since $\sum a_n$ is conv.
Thus by Limit Comp. Test, $\sum a_n^2$ is also conv.

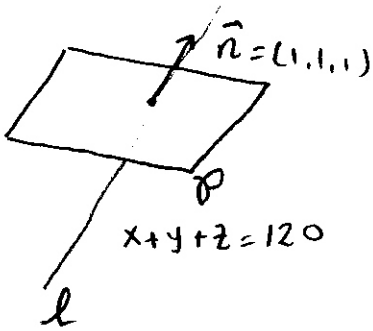
b) Give an example with explanations of a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} a_n^2$ is divergent.

consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is convergent by Alternating series Test

but $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n}}\right)^2 = \sum \frac{1}{n}$ is divergent (harmonic series)

Question 4 (4+4+4+4=16 points)

a) Find parametric equations of the line passing through the point $P_0 = (1, 0, -2)$ and perpendicular to the plane $x + y + z = 120$.



$$\ell = \underset{(1,0,-2)}{P_0} \oplus \underset{\vec{n}=(1,1,1)}{\vec{v}}$$

Thus $\ell : \begin{cases} x = 1 + 1t \\ y = 0 + 1t \\ z = -2 + 1t \\ t \in \mathbb{R} \end{cases}$

b) Find the point of intersection of the line and the plane given in part (a).

$P = (x, y, z) \in \ell \cap \mathcal{P} \Rightarrow P \in \ell \Rightarrow \begin{cases} x = 1 + t \\ y = t \\ z = -2 + t \end{cases}$ for some $t \in \mathbb{R}$

also, $P \in \mathcal{P}$, hence

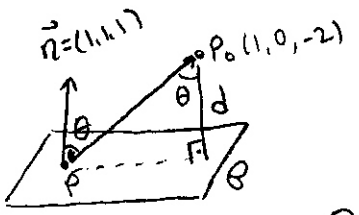
$$x + y + z = (1 + t) + (t) + (-2 + t) = 3t - 1 = 120 \Rightarrow 3t = 121 \Rightarrow t = \frac{121}{3}$$

$$\Rightarrow P = \left(1 + \frac{121}{3}, \frac{121}{3}, -2 + \frac{121}{3}\right) = \left(\frac{124}{3}, \frac{121}{3}, \frac{115}{3}\right)$$

c) Find the distance between the point P_0 and the plane in part (a).

Distance between P_0 and the plane is the distance between P_0 and the intersection point P found in part (b). Hence $d = \sqrt{\left(1 - \frac{124}{3}\right)^2 + \left(\frac{121}{3}\right)^2 + \left(-2 - \frac{115}{3}\right)^2} = \frac{121\sqrt{3}}{3}$

OR

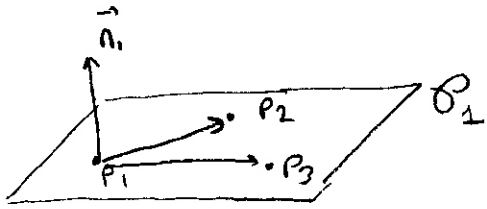


P is any point on $\mathcal{P}$

$$\sin\theta = \frac{d}{\|\vec{PP_0}\|} \Rightarrow d = \|\vec{PP_0}\| \sin\theta = \frac{\|\vec{PP_0}\| \, \|\vec{n}\| \sin\theta}{\|\vec{n}\|}$$

$$\Rightarrow d = \frac{\|\vec{PP_0} \times \vec{n}\|}{\|\vec{n}\|} = \cdots = \frac{121\sqrt{3}}{3}$$

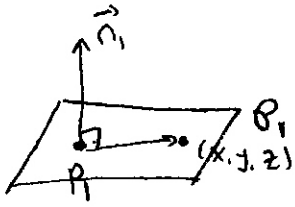
d) Find an equation of the plane containing the points $P_1(39, 40, 41)$, $P_2(40, 40, 40)$ and $P_3(41, 38, 41)$.



$\vec{P_1P_2} = \langle 1, 0, -1 \rangle$, $\vec{P_1P_3} = \langle 2, -2, 0 \rangle$

$$\vec{n}_1 \parallel \vec{P_1P_2} \times \vec{P_1P_3} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 2 & -2 & 0 \end{vmatrix} = \langle -2, -2, -2 \rangle$$

so we can take $\vec{n}_1 = \langle 1, 1, 1 \rangle$



$(x, y, z) \in \mathcal{P}_1 \iff (x - 39, y - 40, z - 41) \cdot (1, 1, 1) = 0$

ie, $x + y + z = 39 + 40 + 41$

$x + y + z = 120$

Question 5 (4 points)
Find the length of the space curve $\vec{r} = \langle 2e^t \cos t, 2e^t \sin t, e^t \rangle$ for $0 \le t \le 1$.

$$r'(t) = \langle 2e^t\cos t - 2e^t\sin t,\ 2e^t\sin t + 2e^t\cos t,\ e^t \rangle$$
$$= e^t \langle 2(\cos t - \sin t),\ 2(\cos t + \sin t),\ 1 \rangle$$
$$\Rightarrow \|r'(t)\|^2 = 4e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t) + 4e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t) +$$
$$= e^{2t}(8+1) = 9e^{2t}$$
$$\Rightarrow \|r'(t)\| = 3e^t$$
$$\Rightarrow L = \int_0^1 3e^t\,dt = 3e^t\Big|_0^1 = 3(e-1) \text{ units.}$$

Question 6 (10 points) Find the Maclaurin Series of $F(x) = \int_0^x \sin(t^2)dt$ and use it to approximate $F(1)$ with $|error| < 10^{-4}$.

Recall that $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \quad \forall t \in \mathbb{R}$

$$\Rightarrow \sin(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}$$
$$\Rightarrow \int_0^x \sin(t^2)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} = F(x)$$
$$\Rightarrow F(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)}$$ is an alternating series with

$b_n = \frac{1}{(2n+1)!\,(4n+3)}$. Clearly b_n is decreasing and $\lim_{n\to\infty} b_n = 0$

So
$$|F(1) - S_n| < b_{n+1} = \frac{1}{(2n+3)!\,(4n+7)} < 10^{-4}$$
$$\Rightarrow (2n+3)!\,(4n+7) > 10^4 = 10.000$$

$n=1 \Rightarrow 5!\cdot 11 = 120\cdot 11 = 1320$

$n=2 \Rightarrow 7!\cdot 15 = (5040)(15) > 10.000$

Hence $F(1) \approx S_2 = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320}$

$|\text{error}| < 5040\cdot 15 < 10^{-4}$

M E T U
Department of Mathematics

Group	CALCULUS II MidTerm 2	Exam Place.

Code : *Math 120* Acad. Year : *2007-2008* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *May.10.2008* Time : *09:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 4 PAGES TOTAL 80 POINTS

1	2	3	4	5	6	7	8	Show Your Work	

Question 1 (10 points) Let $f(x,y) = 1 - 3x^2 - xy$.

a)Find the maximum value of rate of change of f at the point $(1,2)$.

we are asked to find $\max\limits_{\vec{v}\text{ - unit vector}} D_{\vec{v}} f(1,2)$.

Since $D_{\vec{v}} f(1,2) = \nabla f(1,2) \cdot \vec{v} = \|\nabla f(1,2)\| \cos\theta$ where θ is the angle betw the unit vector $\vec{v}$ and vector $\nabla f(1,2)$.

Since $\cos\theta \leq 1$, Maximum value is $\|\nabla f(1,2)\|$ (for $\vec{v} = \frac{\nabla f(1,2)}{\|\nabla f(1,2)\|}$)

$$\left.\begin{array}{l} f_x(x,y) = -6x - y \Rightarrow f_x(1,2) = -8 \\ f_y(x,y) = -x \Rightarrow f_y(1,2) = -1 \end{array}\right\} \Rightarrow \|\nabla f(1,2)\| = \|(-8,-1)\| = \sqrt{64+1} = \sqrt{65}$$

b)Find all directions (unit vectors) in which rate of change of f at the point $(1,2)$ is zero.

Rate of change, that is directional derivative, is zero if $\cos\theta =$ that is if $\nabla f(1,2) = (-8,-1)$ is perpendicular to $\vec{v}$.

(ie, $\nabla f(1,2) \cdot \vec{v} = 0$)

Thus $\vec{v} \in \left\{ \frac{1}{\sqrt{65}}(1,-8), \frac{1}{\sqrt{65}}(-1,8) \right\}$

Question 2 (10 points) Given the equation $x + y + z = e^z$, compute $z_{yx} = \dfrac{\partial^2 z}{\partial x \partial y}$ at the point P_0 where $z(P_0) = 120$.

$x+y+z = e^z$, differentiating both sides wrt y;

$1 + z_y = z_y e^z$ ($\Rightarrow$ at P_0, $z_y(P_0) = \frac{1}{e^{120}-1}$) also $z_x(P_0) = \frac{1}{e^{120}}$

now wrt x;

$z_{yx} = z_{yx} e^z + z_y z_x e^z$

thus $z_{yx}(P_0) = \dfrac{z_y(P_0)\, z_x(P_0)\, e^{z(P_0)}}{1 - e^{z(P_0)}} = \dfrac{\left(\frac{1}{e^{120}-1}\right)^2 e^{120}}{1 - e^{120}}$

M E T U
Department of Mathematics

Group	CALCULUS II MidTerm 2	Exam Place.

Code : *Math 120* Acad. Year : *2007-2008* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *May.10.2008* Time : *09:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 4 PAGES TOTAL 80 POINTS

1	2	3	4	5	6	7	8	Show Your Work	

Question 1 (10 points) Let $f(x,y) = 1 - 3x^2 - xy$.

a) Find the maximum value of rate of change of f at the point $(1,2)$.

We are asked to find $\underset{\vec{v}\text{ - unit vector.}}{\text{Max}}\ D_{\vec{v}} f(1,2)$.

since $D_{\vec{v}} f(1,2) = \nabla f(1,2) \cdot \vec{v} = \|\nabla f(1,2)\| \cos\theta$ where θ is the angle between the unit vector $\vec{v}$ and vector $\nabla f(1,2)$.

since $\cos\theta \leq 1$, Maximum value is $\|\nabla f(1,2)\|$ $\left(\text{for } \vec{v} = \dfrac{\nabla f(1,2)}{\|\nabla f(1,2)\|}\right)$

$$\left.\begin{array}{l} f_x(x,y) = -6x - y \Rightarrow f_x(1,2) = -8 \\ f_y(x,y) = -x \Rightarrow f_y(1,2) = -1 \end{array}\right\} \|\nabla f(1,2)\| = \|(-8,-1)\| = \sqrt{64+1} = \sqrt{65}$$

b) Find all directions (unit vectors) in which rate of change of f at the point $(1,2)$ is zero.

Rate of change, that is directional derivative is zero if $\cos\theta = 0$, that is if $\nabla f(1,2) = (-8,-1)$ is perpendicular to $\vec{v}$.

Thus $\vec{v} \in \left\{ \frac{1}{\sqrt{65}}(1,-8),\ \frac{1}{\sqrt{65}}(-1,8) \right\}$

Question 2 (10 points) Given the equation $x + y + z = e^z$, compute $z_{yx} = \dfrac{\partial^2 z}{\partial x \partial y}$ at the point P_0 where $z_x(P_0) = 251$, $z_y(P_0) = 8$ and $z(P_0) = 120$.

I) gets full credit

$x+y+z = e^z$, differentiate both sides wrt y;

$1 + z_y = z_y e^z$, now wrt x;

$z_{yx} = z_{yx} e^z + z_y z_x e^z$

Hence

$z_{yx} = \dfrac{z_y z_x e^z}{1 - e^z}$. At P_0 we have $z_{yx} = \dfrac{251 \cdot 8 \cdot e^{120}}{1 - e^{120}} = \dfrac{2008\, e^{120}}{1 - e^{120}}$

II) gets full credit

$z_y = \dfrac{1}{e^z - 1} \Rightarrow z_{yx} = \dfrac{-z_y e^z}{(e^z-1)^2} \Rightarrow z_{yx}(P_0) = -\dfrac{e^{120} \cdot 251}{(e^{120}-1)^2}$

III) gets full credit

Notice that there is a mistake in this question. Since $z_x(P_0) = 251 \neq 8 = z_y(P_0)$ can not be satisfied. $z_x = \dfrac{1}{e^z - 1} = z_y$, given conditions

If you noticed this, and explained in your paper, you will also get full credit.

Question 3 (10 points) Given $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$, by $f(x,y) = \dfrac{xy}{x^2+y^2}$. Can you define $f(0,0)$ so that f is continuous everywhere. Explain.

Since f is a polynomial over polinomial, it is continuous everywhere except the zeros of denominator, that is at $(0,0)$. To define $f(0,0)$ so that f is also continuous at this point, we must check $\lim_{(x,y)\to(0,0)} f(x,y)$. If this limit exists, define $f(0,0)$ to be this limit and hence f is continuous everywhere. If limit does not exist, we can not make f continuous at $(0,0)$.

Notice that lines given by $y = mx$ pass through $(0,0)$ and along these lines, limit is $\lim_{\substack{x\to 0\\(x\neq 0)}} f(x,mx) = \lim_{x\to 0} \dfrac{xmx}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$

Since result depends on m, limit does not exist. Hence we can NOT define $f(0,0)$ so that f is continuous everywhere

Question 4 (10 points) Find the length of the polar curve $r = f(\theta) = 1 + \cos(\theta)$ for $\theta \in [0, 2\pi]$.

Recall that $\ell = \int_0^{2\pi} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$ where $f(\theta) = 1+\cos\theta$

$f'(\theta) = -\sin\theta$

$f^2 + f'^2 = 2 + 2\cos\theta$

So

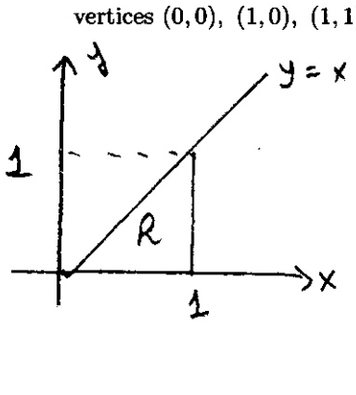
$$\ell = \int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{4\cos^2\left(\tfrac{\theta}{2}\right)}\, d\theta = \int_0^{2\pi} 2\left|\cos\tfrac{\theta}{2}\right| d\theta$$

$$= 2\left[\int_0^{\pi} \cos\tfrac{\theta}{2}\, d\theta - \int_{\pi}^{2\pi} \cos\tfrac{\theta}{2}\, d\theta\right] = 2\left[2\sin\tfrac{\theta}{2}\Big|_0^{\pi} - 2\sin\tfrac{\theta}{2}\Big|_{\pi}^{2\pi}\right]$$

$$= 4\big((1-0)-(0-1)\big) = 4\cdot 2 = 8$$

(or, since $f(-\theta) = 1+\cos(-\theta) = 1+\cos(\theta) = f(\theta)$, $\ell = 2\int_0^{\pi} \sqrt{f^2+f'^2}\, d\theta$)

Question 5 (10 points) Evaluate $\int_R\int \dfrac{xy}{1+x^4}\, dA$ where R is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.



$$= \int_0^1 \int_0^x \frac{x}{1+x^4}\, y\, dy\, dx = \int_0^1 \frac{x}{1+x^4} \frac{y^2}{2}\Big|_{y=0}^{y=x} dx$$

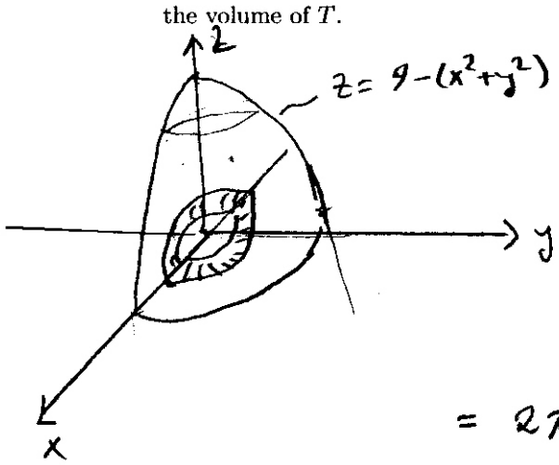
$$= \frac{1}{2}\int_0^1 \frac{x^3}{1+x^4}\, dx = \frac{1}{2}\frac{1}{4}\int_1^2 \frac{1}{u}\, du = \frac{1}{8}\ln|u|$$

$u = 1+x^4$

$du = 4x^3\, dx$

$$= \frac{1}{8}\ln 2$$

Question 6 (10 points) Consider the solid T that lies between two cylinders $x^2+y^2=1$, $x^2+y^2=4$ and bounded by the surfaces $z=0$, $z=9-(x^2+y^2)$. Compute the volume of T.



$$V = \iint 9-(x^2+y^2)\, dy\, dx$$

use Polar coord's

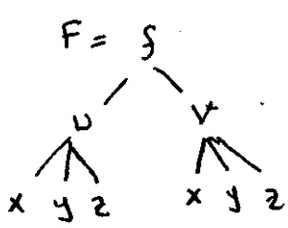
$$= \int_0^{2\pi}\int_1^2 (9-r^2)\, r\, dr\, d\theta$$

$$= 2\pi \left(\frac{9r^2}{2} - \frac{r^4}{4}\right)_1^2 = 2\pi\left((18-4) - \left(\frac{9}{2} - \frac{1}{4}\right)\right)$$

$$= 2\pi\left(14 - \frac{17}{4}\right) = \frac{39}{2}\pi$$

Question 7 (10 points) Let $f(u,v)$ be a differentiable function such that $f(-1,3)=0$, $f_u(-1,3)=2$, $f_v(-1,3)=-5$. Find an equation of the plane tangent to the surface defined by $f(xyz, x^2+y^2+z^2)=0$ at the point $P_0(1,-1,1)$.

Let $F(x,y,z) = f(xyz,\ x^2+y^2+z^2)$. Then given surface is the level surface of F (given by $F(x,y,z)=0$) and hence $\nabla F(P_0)$ is normal to the tangent plane to the surface at P_0.



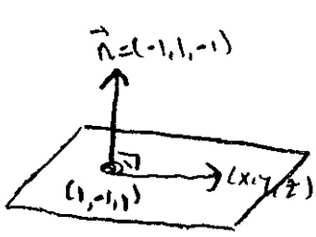
where $u = xyz$, $v = x^2+y^2+z^2$

$$F_x = f_u(u,v)\, u_x(x,y,z) + f_v \cdot v_x$$
$$= f_u(-1,3)\, u_x(1,-1,1) + f_v(-1,3)\, v_x(1,-1,1)$$
$$= 2\cdot(-1) + (-5)\,2 = -12$$

$$F_y = f_u u_y + f_v v_y = 12$$
$$f_z = f_u u_z + f_v v_z = -12$$

Thus $\nabla F(P_0) = (-12, 12, -12)$

so we can take $\vec{n} = (-1,1,-1)$ $\quad (\parallel \nabla F(P_0))$



$$(x,y,z) \in \mathcal{P} \iff (x-1, y+1, z-1)\cdot(-1,1,-1) = 0$$

That is

$$-x+1+y+1-z+1 = 0$$

$$\boxed{x - y + z = 3}$$

Question 8 (10 points) Let $f(x,y) = 2 + 2x + 2y - x^2 - y^2$.

a)Find all critical points of f and classify them as a local maximum, a local minimum or a saddle point.

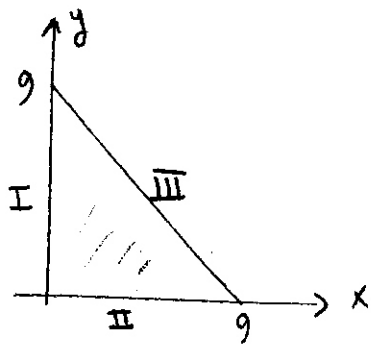
$f_x = 2 - 2x = 0$
$f_y = 2 - 2y = 0$ $\Rightarrow (x,y) = (1,1)$ is the only critical point.

$f_{xx} = -2$
$f_{yy} = -2$
$f_{xy} = 0$ at $(1,1)$ we have $f_{xx} f_{yy} - f_{xy}^2 = 4 > 0$

$f_{xx} = -2 < 0$

so $f(1,1) = 4$ is a local maximum point.

b)Find the absolute (global) maximum and absolute minimum values of f on the triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.



1-) interior points;

from part (a), $(1,1)$ is the only critical point and $\boxed{f(1,1) = 4}$

2-) boundary points

(i) points on I : $x = 0$, $y \in [0,9]$

$f = 2 + 2y - y^2$ on $[0,9]$

$f' = 2 - 2y = 0 \Rightarrow y = 1$ critical point

$\boxed{f(0,1) = 3}$ $\boxed{f(0,0) = 2}$ $\boxed{f(0,9) = -61}$

boundary points

(ii) Points on II : $y = 0$, $x \in [0,9]$

$f = 2 + 2x - x^2$; $x \in [0,9]$

$f' = 2 - 2y = 0 \Rightarrow x = 1$ critical point

$\boxed{f(1,0) = 3}$ $\boxed{f(0,0) = 2}$ $\boxed{f(9,0) = -61}$

(iii) Points on III : $y = 9 - x$, $x \in [0,9]$

$\Rightarrow f = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$= -61 + 18x - 2x^2 \Rightarrow f' = 18 - 4x = 0 \rightarrow x = 9/2$ critical point

$\boxed{f(\frac{9}{2}, \frac{9}{2}) = -\frac{41}{2}}$ $\boxed{f(0,9) = -61}$ $\boxed{f(9,0) = -61}$

Thus Absolute MAX is $f(1,1) = 4$

Absolute min is $f(9,0) = f(0,9) = -61$

M E T U
Department of Mathematics

Group	CALCULUS II Final Exam	Exam Place.

Code : *Math 120* Acad. Year : *2007-2008* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *June.05.2008* Time : *09:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 4 PAGES TOTAL 90 POINTS

1	2	3	4	5	6	7	Show Your Work	

Question 1 (8+4=12 points)

a) Find the function $f(x)$ whose Maclaurin Series is $\sum_{n=0}^{\infty} x^{3n} - \sum_{n=0}^{\infty} x^{3n+1}$. Specify domain of f.

Recall that if $|x|<1$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Thus $\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$ $\forall\ |x^3|<1$ ie, $\forall\ |x|<1$.

$$\sum_{n=0}^{\infty} x^{3n} - x\sum_{n=0}^{\infty} x^{3n} = (1-x)\sum_{n=0}^{\infty} x^{3n} = (1-x)\frac{1}{1-x^3}$$

Hence $f(x) = \frac{(1-x)}{1-x^3}$ for $x \in (-1, +1)$.

$$= \frac{1}{1+x+x^2}$$

b) For the function in part (a), compute the 120th derivative $f^{(120)}(0)$.

Recall that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \cdots + \frac{f^{(120)}(0)}{120!} x^{120} + \cdots$

Thus $\frac{f^{(120)}(0)}{120!}$ is coefficient of x^{120} in the series expansion of $f(x)$.

$$(1-x)\sum_{n=0}^{\infty} x^{3n} = (1-x) + (1-x)x^3 + \cdots + (1-x)x^{120} + \cdots = 1 - x + x^3 - x^4 + \cdots + x^{1}$$

Thus $\frac{f^{(120)}(0)}{120!} = 1 \implies f^{(120)}(0) = 120!$

Question 2 (6+6=12 points) Determine whether the followings are convergent or divergent. Explain.

a) $\sum_{n=1}^{\infty} \frac{n^{120}+2008}{120^n} = \sum_{n=1}^{\infty} a_n$, let $b_n = \frac{n^{120}}{120^n}$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, by limit comparison theorem, we have $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent

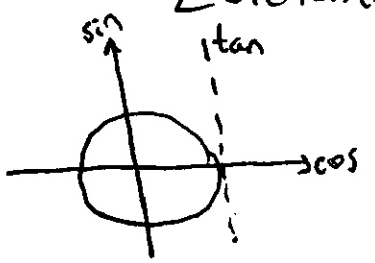
Apply Ratio Test to b_n $(b_n > 0)$

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)^{120}}{120^{\cancel{n+1}}} \cdot \frac{\cancel{120^n}}{n^{120}} = \lim_{n\to\infty} \frac{1}{120}\left(\frac{n+1}{n}\right)^{120} = \frac{1}{120} < 1$$

By Ratio Test $\sum b_n$ is convergent, and hence $\sum a_n$ is convergent

b) $\sum_{n=120}^{\infty} \arctan(n)$. Since $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$, by n^{th} Term Test,

$\sum \arctan(n)$ is divergent



Question 3 (10 points) Find $\vec{\nabla} f(a,b)$ for a differentiable function $f(x,y)$ given by the directional derivatives $D_{\vec{u}} f(a,b) = 3\sqrt{2}$ and $D_{\vec{v}} f(a,b) = 5$ where $\vec{u}$ is the unit vector in the direction of the vector $\left\langle \frac{2008}{\sqrt{120}}, \frac{2008}{\sqrt{120}} \right\rangle$ and $\vec{v}$ is the unit vector in the direction of the vector $\left\langle \frac{360}{\sqrt{120}}, \frac{-480}{\sqrt{120}} \right\rangle$.

Note that $\left\langle \frac{2008}{\sqrt{120}}, \frac{2008}{\sqrt{120}} \right\rangle \parallel \langle 1,1 \rangle$, hence $\vec{u} = \frac{1}{\sqrt{2}}\langle 1,1 \rangle$

$\left\langle \frac{360}{\sqrt{120}}, \frac{-480}{\sqrt{120}} \right\rangle \parallel \langle 3,-4 \rangle$, hence $\vec{v} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

at the point (a,b) we have (using $D_{\vec{w}} f(a,b) = \nabla f(a,b) \cdot \vec{w}$)

$$\frac{1}{\sqrt{2}} f_x + \frac{1}{\sqrt{2}} f_y = 3\sqrt{2} \implies f_x + f_y = 6$$

and

$$\frac{3}{5} f_x - \frac{4}{5} f_y = 5 \implies 3f_x - 4f_y = 25$$

$$\implies f_x(a,b) = 7, \quad f_y(a,b) = -1$$

Thus $\nabla f(a,b) = \langle 7, -1 \rangle$

Question 4 (14 points) Find the maximum and minimum values of the function $f(x,y,z) = x+y+z$ on the ellipsoid $x^2+4y^2+9z^2 = 1764$.

Use Lagrange Multipliers Method to find Max/min of $f(x,y,z) = x+y+z$ subject to $g(x,y,z) = x^2+4y^2+9z^2 = 1764$

set $\nabla f = \lambda \nabla g$

$$\left.\begin{aligned} 1 &\overset{(1)}{=} 2\lambda x \\ 1 &\overset{(2)}{=} 8\lambda y \\ 1 &\overset{(3)}{=} 18\lambda z \\ x^2+4y^2+9z^2 &\overset{(4)}{=} 1764 \end{aligned}\right\}$$

4 equations
4 unknowns
☺

$(1) \Rightarrow \lambda \neq 0$

$(2,3) \Rightarrow x = \frac{1}{2\lambda}, \quad y = \frac{1}{8\lambda}, \quad z = \frac{1}{18\lambda}$

$(4) \Rightarrow \frac{1}{4\lambda^2} + \frac{1}{16\lambda^2} + \frac{1}{36\lambda^2} = 1764$

$\Rightarrow \frac{49}{144\lambda^2} = 1764 = 42^2$

$\Rightarrow \lambda = \mp \frac{1}{72}$

$\Rightarrow x = \mp 36, \quad y = \mp 9, \quad z = \mp 4$

$\Rightarrow$ Max of $f(x,y,z)$ is $f(36, 9, 4) = 49$

min of $f(x,y,z)$ is $f(-36, -9, -4) = -49$

Question 5 (14 points) Evaluate $\int_C 6x^2y\,dx + (2x^3-2y)\,dy$ where C is the piece of the curve $x = \cos(\pi y)$ from the point $P_0(1,0)$ to the point $P_1(0,\frac{1}{2})$.

Let $\left.\begin{aligned} M(x,y) &= 6x^2y \\ N(x,y) &= 2x^3-2y \end{aligned}\right\} \Rightarrow M_y = 6x^2 = N_x$

$\Rightarrow \int_C M\,dx + N\,dy$ is independant of path. it only depends on the end points of the curve C, that is $\int_C M\,dx + N\,dy = f(P_1) - f(P_0)$

where $f(x,y)$ is a potential function for the vector field $F = \langle M,$

$F = \nabla f \Rightarrow f_x = M = 6x^2y \Rightarrow f = 2x^3y + c(y)$

$\Rightarrow f_y = 2x^3 + c'(y) = N$
$= 2x^3 - 2y$

$\Rightarrow c'(y) = -2y \Rightarrow c(y) = -y^2 + c$

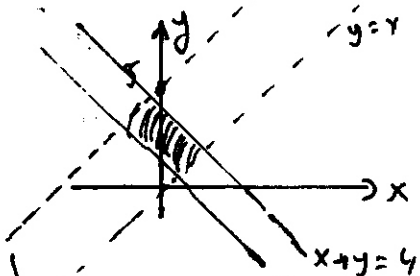
Thus

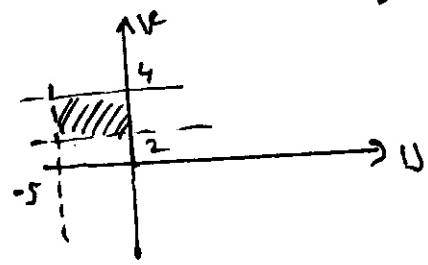
$f(x,y) = 2x^3y - y^2 + c$

and $\int_C F \cdot dr = \int_C M\,dx + N\,dy = f(P_1) - f(P_0) = (c) - (-\frac{1}{4} + c) = \frac{1}{4}$

Question 6 (14 points) Evaluate $\int_R\int \frac{e^{x-y}}{(x+y)}dA$. Where R is the rectangle bounded by the lines $y=x$, $y=x+5$, $y=2-x$, and $y=4-x$.

Let $u = x-y$, $v = x+y$ $\Big\}$ $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$ $\Big\}$ $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = 1/2$





$x+y=4 \leftrightarrow v=4$

$x+y=2 \leftrightarrow v=2$

$x-y=0 \leftrightarrow u=0$

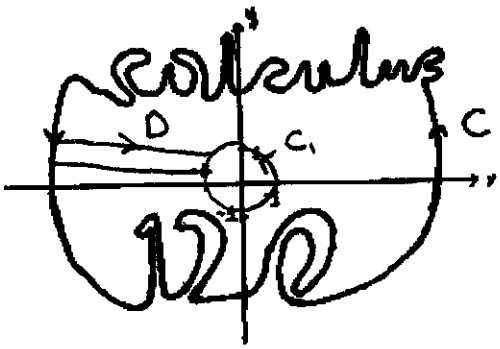
$x-y=-5 \leftrightarrow u=-5$

$$\iint_{R_{xy}} \frac{e^{x-y}}{x+y}\,dy\,dx = \iint_{R_{uv}} \frac{e^u}{v}\,\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_2^4\int_{-5}^0 \frac{e^u}{v}\,du\,dv = \frac{1}{2}\int_2^4 \frac{1}{v}(e^0 - e^{-5})\,dv = \frac{1}{2}(1-e^{-5})(\ln$$

$$= \frac{1}{2}\left(1-\frac{1}{e^5}\right)\ln 2.$$

Question 7 (14 points) The curve shown encloses an area of 24π. Evaluate the integral $\oint_C (3x + \frac{2x}{x^2+y^2})dx + (5x + \frac{2y}{x^2+y^2})dy$.



Let $M(x,y) = 3x + \frac{2x}{x^2+y^2}$, $N(x,y) = 5x + \frac{2y}{x^2+y^2}$

Then $N_x - M_y = 5$.

$$\oint_{C+D-C_1-D} M\,dx + N\,dy = \iint_R 5\,dA = 5\,\text{Area}(R) = 5\,(24\pi - \underbrace{\pi}_{\text{area enclosed by}}) = 115\pi$$

closed curve NOT enclosing $(0,0)$.

Thus $\oint_C M\,dx + N\,dy = 115\pi + \oint_{C_1} M\,dx + N\,dy = 115\pi + \int_0^{2\pi}[(3\cos t + 2\cos t)(-\sin$

$C_1 \leftrightarrow r(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$+ (5\cos t + 2\sin t)(\cos t)]$

$= 115\pi + 5\pi = \underline{\underline{120\pi}}$